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Sufficiency Conditions for a Commonly Used Downstream Boundary Condition on Stream Function

Consider the numerical calculation of a boundary layer over a flat plate in planar incompressible flow, using the full Navier–Stokes equations in the stream function-vorticity form and a regular, rectangular mesh. The geometry and notation are indicated in Fig. 1. This problem is representative of many numerical fluid-dynamics problems in which the analytical boundary conditions should be applied at an infinite distance from the region of interest, but in which the computational boundary conditions must be applied at the limits of the computational mesh.[1]

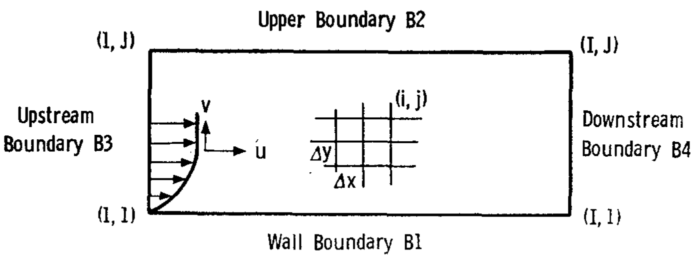

FIG. 1. Geometry of the problem

We are concerned with the commonly used [1–3] technique of determining the downstream boundary condition on the stream function ψ by setting $\delta v/\delta x = 0$ (where $\delta/\delta x$ is the usual centered difference analogue of $\partial/\partial x$) near the downstream boundary B4. There is no particular physical justification for this boundary condition. It is simply the least restrictive method found by numerical experimentation which does not lead to obvious drifting or to more catastrophic failure of the computations. It is clear that, if this condition is sufficient to provide a unique solution, then the computation will yield a valid approximation to the physical flow, provided that the boundary B4 is far enough from the region of interest. Just how far is "far enough" obviously depends on the particular problem, and cannot be answered in general.

[1] We do not use the words "analytical" and "computational" with any suggestion of "exact" and "approximate." A mature view of physics demands that both the analytical and the computational approaches be recognized as approximate.

Since ψ is related to the velocity components u and v by $\delta\psi/\delta x = -v$ and $\delta\psi/\delta y = u$, the linear extrapolation

$$\psi(I,j) = 2\psi(I-1,j) - \psi(I-2,j) \tag{1}$$

thus sets $\delta^2\psi/\delta x^2 = -\delta v/\delta x = 0$ at $(I-1,j)$. (We may alternately consider the computational mesh to end with B4 located at $I-1$, and the points at I to be "fictitious" points defined for the convenience of using regular interior point differencing for ψ at $I-1$.) The extrapolation is performed at every iterative sweep in the solution of the discretized Poisson equation,

$$\delta^2\psi/\delta x^2 + \delta^2\psi/\delta y^2 = \zeta. \tag{2}$$

The vorticity ζ is separately determined at interior points during each iterative sweep in the solution of the vorticity transport equation. The question is, does this technique (1) of linear extrapolation provide a sufficient boundary condition to determine a unique solution in Eq. (2)?

We consider Dirichlet boundary conditions along B1 with $\psi(i, 1) = 0$, and along B3, where $\psi(1,j) = q(j)$, some specified inflow function. Along the upper boundary B2, three types of boundary conditions are considered. The first, a Dirichlet condition,

$$\psi(i, J) = \psi(1, J) \tag{3}$$

makes B2 a streamline, analogous to a wind-tunnel wall. (The "wall" may be made an inviscid wall by proper treatment of the vorticity boundary condition [1–4]). The second condition

$$\psi(i, J) = U \times \Delta y - \psi(i, J-1) \tag{4}$$

approximates the Neumann condition $\partial\psi/\partial y = U$, with B2 located at $j = J - \frac{1}{2}$. It physically corresponds to fixing the u-component of velocity at the "free stream" value U, while allowing the v-component to develop as part of the solution. The third condition is a plausible analogy of the downstream boundary condition,

$$\psi(i, J) = 2\psi(i, J-1) - \psi(i, J-2). \tag{5}$$

Consider a one-dimensional continuum version of the problem.

$$\frac{d^2\psi}{dx^2} = \zeta, \qquad \text{with } \psi = \psi(1) \text{ at B3}, \qquad \frac{d^2\psi}{dx^2} = 0 \text{ at B4}. \tag{6}$$

It is obvious that the second boundary condition, corresponding to the linear extrapolation (1), is either contradictory to the differential equation if $\zeta \neq 0$ at B4;

or, for $\zeta = 0$ at B4, it merely restates the differential equation, i.e., it is no boundary condition. Thus (6) is not a complete problem and, in the finite-difference formulation, linear extrapolation fails to determine a unique solution, in the one-dimensional case.

Now, consider the two-dimensional continuum problem. The condition $\partial^2\psi/\partial x^2 = 0$ at B4 reduces the Poisson equation to

$$d^2\psi/dy^2 = \zeta \text{ at B4.} \tag{7}$$

For downstream conditions (3) or (4), this constitutes a two-point boundary-value problem with Dirichlet conditions at B1, and Dirichlet (3) or Neumann (4) conditions at B2. Since these are known to provide a unique solution to (7), in both its continuum and finite-difference forms, it follows that ψ at B4 is uniquely determinable. Thus, the extrapolation procedure (1) would be analogous to a Dirichlet condition at B4, which is known to be sufficient. Note, however, that the upper boundary condition (5) is not sufficient to determine the problem. As in the one-dimensional case, it either contradicts the partial differential equation, if $\zeta \neq 0$ at B2, or simply restates the differential equation if $\zeta = 0$ at B2. The sufficiency of the downstream boundary condition (1) is thus seen to depend on the conditions used at adjacent boundaries, indicating the significance of dimensionality to the problem.

Note that the numerical problem using (1) and (5) might "converge" to within some specified tolerance, or that discretization could conceivably make the solution unique, i.e., independent of the initial estimate. But that unique solution will depend on Δx and Δy and, as Δx, $\Delta y \to 0$, the problem would become indeterminate.

The above arguments also suggest an efficient method of implementing the condition $\delta^2\psi/\delta x^2 = 0$ near the downstream boundary. Instead of applying it at $i = I - 1$ by linear extrapolation, it may be applied at $i = I$ directly, reducing (2) to the discretized ordinary differential equation $\delta^2\psi/\delta y^2 = \zeta$, with the two-point boundary conditions of $\psi(I, 1) = 0$ at B1 and either (3) or (4) at B2. The value of $\zeta(I, j)$ at B4 can be determined by several methods [4] for downstream computational boundary conditions on vorticity.[2] This ordinary difference equation can then be quickly solved noniteratively by the tridiagonal matrix algorithm [5] (see Appendix). With the downstream boundary values of ψ so determined, one can proceed confidently with the solution of the partial difference Eq. (2), with Dirichlet conditions at B4.

[2] Maltreatment of the ζ condition at B4 may cause the entire coupled ζ, ψ computation (of which the above Poisson equation is merely a nested problem) to drift. In particular, linear extrapolation on ζ and ψ does not work [2]. But a zero gradient condition, $\zeta(I, j) = \zeta(I - 1, j)$, does work, and if vortex shedding does not occur, more accurate methods are available [4].

The arguments and the above implementation are also extendible to dimensions greater than 2, although ψ is no longer interpretable as a stream function.

APPENDIX

The general method of solution of a tridiagonal matrix may be found in [5]. For the particularly simple one-dimensional discretized Poisson equation used here,

$$\frac{\psi(I,j+1) - 2\psi(I,j) + \psi(I,j-1)}{\Delta y^2} = \zeta(I,j), \tag{A.1}$$

the following noniterative shooting method is recommended. We have $\psi(I, 1) = 0$. Choose a provisional $\psi'(I, 2) = 0$. This is in error from the true value $\psi(I, 2)$ by the amount e, that is,

$$\psi(I, 2) = \psi'(I, 2) + e. \tag{A.2}$$

The remaining provisional values up through J are calculated in one sweep, starting at $j = 3$, by rearrangement of (A.1).

$$\psi'(I,j+1) = \zeta(I,j)\,\Delta y^2 + 2\psi'(I,j) - \psi'(I,j-1). \tag{A.3}$$

The unit error e may be shown to propagate as

$$\psi(I,j) = \psi'(I,j) + (j-1)\,e. \tag{A.4}$$

The unit error e is calculated from $\psi'(I, J)$ and the boundary condition at $B2$. If the Dirichlet condition (3) is used,

$$e = \frac{\psi(I, J) - \psi'(I, J)}{J - 1} \tag{A.5}$$

and if the Neumann condition (4) is used,

$$e = U \times \Delta x - \psi'(I, J) + \psi'(I, J - 1). \tag{A.6}$$

In the second sweep, the $\psi'(I,j)$ are corrected to $\psi(I,j)$ by application of (A.4). Since (A.4) shows that the total error propagates linearly in j, there is no danger of computer round-off errors destroying the accuracy, as sometimes occurs with shooting methods.

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